Energy dynamics in the Sinai model

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We study the time dependent potential energy $W(t) = U(x(0)) - U(x(t))$ of a particle diffusing in a one dimensional random force field (the Sinai model). Using the real space renormalization group method (RSRG), we obtain the exact large time limit of the probability distribution of the scaling variable $w = W(t)/(T \ln t)$. This distribution exhibits a *nonanalytic* behaviour at $w = 1$. These results are extended to a small non-zero applied field. Using the constrained path integral method, we moreover compute the joint distribution of energy $W(t)$ and position $x(t)$ at time t. In presence of a reflecting boundary at the starting point, with possibly some drift in the + direction, the RSRG very simply yields the one time and aging two-time behavior of this joint probability. It exhibits differences in behaviour compared to the unbounded motion, such as analyticity. Relations with some magnetization distributions in the 1D spin glass are discussed.

I. INTRODUCTION

The description of glassy activated dynamics is one of the challenges of the present theory of glasses and disordered systems [1,2]. In fully connected, mean field approximations of these systems, the energy initially decreases and saturates at a threshold: the asymptotic large time dynamics then remains entirely dominated by high saddles in energy landscape and not by barriers. A recently studied model [3], still mean field in character but which incorporates some effects of finite connectivity exhibits an additional regime of late times with subthreshold dynamics. It mimics an activated dynamics between lower and lower energy states, expected in any realistic physical system, leading to an interesting, slow but unbounded, decrease of the energy.

At the other end of the spectrum, the Sinai model [4] of a particle diffusing in a one dimensional quenched random force field is dominated by activated dynamics and, despite its simplified character, is a non trivial example of a glass. It has the advantage of being analytically tractable, most notably through the real space renormalization group method (RSRG) [10,11] and by various other methods [4–9]. Its extensions to many particles describe a variety of coarsening, domain wall growth and reaction diffusion models with disorder in one dimension, also analytically tractable via the RSRG [16,17,14,15]. It is thus interesting to study, within the Sinai model, how the energy decreases as a function of time, as the particle jumps from deeper to deeper wells.

The aim of this paper is to obtain exact results on the large time behaviour of the energy $E(t)$ as a function of time of a particle in Sinai model. The energy lansdcape, denoted $U(x)$, is itself a random walk with $\overline{(U(x) - U(x'))^2} \sim |x - x'|$ at large scales. Fixing the energy to be zero at initial time, and denoting $x(t)$ the trajectory of a single particle, it is defined as:

$$E(t) = U(x(t)) - U(x(0)) = -W(t) \tag{1}$$

i.e it is minus the loss of potential energy $W(t)$, the quantity on which we focus from now on. In the Sinai model this quantity exhibits non trivial sample to sample fluctuations, even in the large time limit. In this paper, we will thus be interested in the distribution $D_t(W)$ of W over both thermal histories and environments (assuming a uniform probability to start at any site). To our knowledge, this probability distribution of the energy has not been computed previously.

We will first consider the unbiased diffusion and use the RSRG approach (Section II B). This method consists in coarse-graining the energy landscape in a way that exactly preserves the long time dynamics [10,13]. One decimates iteratively the smallest energy barrier in the system stopping when the Arrhenius time to surmount the smallest remaining barrier Γ is of order the time scale of interest t, with $\Gamma = T \ln t$. The large time dynamics becomes asymptotically identical to the so called "effective dynamics" where the particle is with probability one at the bottom of the valley in the renormalized landscape containing the starting point. As we find the energy at this point is such that $W(t) \sim T \ln t$, i e the decrease is only logarithmic in time (much slower than the decay of the energy in the model studied in Ref [3]). The correct scaling variable, which exhibits a universal probability distribution at large time is then:

$$w = \frac{W(t)}{T \ln t} \tag{2}$$

and below, we compute this universal large time distribution $D(w)$. The most notable result, besides the simple, exponential form of this distribution, is that this function is found to be *non analytic* at $w = 1$. For any finite time it is of course analytic, but in the large time limit the distribution of the *rescaled* quantity exhibits a jump in its second derivative. As discussed below this can be traced to the decimation process in the RSRG, but since this is a property which can be measured with no a priori knowledge of the physics of the system, e.g. via a numerical simulation, it provides a rather remarkable "experimental" signature that the glass fixed point describing this system is of the type exactly solvable by RSRG, i.e. characterized by an infinitely broad distribution of barriers. A interesting question is whether this is a more general property: this suggests to search for observables exhibiting similar signatures in other systems with "infinite disorder" glass fixed point [13].

The decrease with time of the energy can be related to the H-theorem for this system. One recalls that the generalized free energy:

$$G(t) = \int dx [TP(x,t) \ln P(x,t) + U(x)P(x,t)] \tag{3}$$

is always decreasing in each sample :

$$\partial_t G(t) = -\int dx \frac{J(x,t)^2}{P(x,t)} \tag{4}$$

a simple consequence of the Fokker Planck equation for the probability distribution $P(x,t)$ of the position of the particle $\partial_t P = -\partial_x J$, with $J = -T\partial_x P - P\partial_x U$. Here one can easily see, since the effective dynamics becomes asymptotically exact, that:

$$\frac{G(t)}{\ln t} \underset{t\to\infty}{\simeq} -\frac{W(t)}{\ln t} \tag{5}$$

confirming the considerations given in [18].

Next, we will study the case of a small additional applied external force, for which the RSRG is still exact, and the result for $D(W)$ will be generalized to that case in Section II C. In the following Section III we use a different and complementary method, the constrained path integrals, to obtain the joint distribution of the position $x(t)$ and energy $W(t)$. This yields information on the distribution of positions corresponding to a fixed potential energy loss, and we show, in particular that this distribution becomes deterministic at large W.

Another situation studied here in Section IV corresponds to the geometry of the half-line, with a reflecting wall at the starting point. Then the RSRG allows to solve not only the joint distribution of position and energy at time t but also the full aging two-time behavior of this joint probability. The resulting physics is slightly different, however, as the distribution of energy does not exhibit the nonanalytic behaviour.

Finally, the quantities studied here also have an interpretation in terms of a spin model, the 1D spin glass. We show in Section V that the distributions of magnetizations of that model on some intervals is related to the above results.

II. DISTRIBUTION OF ENERGY

In this section, we compute the distribution $D_t(W)$ of the energy $W(t)$ (1) by solving the appropriate RSRG equation.

A. Review of the main results from the real space renormalization method

We now briefly recall the RSRG method for Sinai landscape, and we refer the reader to [11] for details. In the RSRG, the original energy landscape is replaced by a renormalized landscape at scale Γ, via an iterative decimation procedure where barriers smaller than Γ are eliminated. This generates a joint distribution $P_\Gamma(F,l)$ of the barrier F and length l for the renormalized bonds in the landscape at scale Γ.

In the symmetric case, the probability distribution of the rescaled variables $\eta = \frac{F-\Gamma}{\Gamma}$ and $\lambda = \frac{l}{\Gamma^2}$ flows towards the fixed-point [11]

$$P^*(\eta, \lambda) = LT_{s\to\lambda}^{-1}\left(\frac{\sqrt{s}}{\sinh\sqrt{s}} e^{-\eta\sqrt{s}\coth\sqrt{s}}\right) \tag{6}$$

In particular, the distribution of the barrier alone is a simple exponential

$$P^*(\eta) = \int_0^{+\infty} d\lambda P^*(\eta, \lambda) = e^{-\eta} \tag{7}$$

In the effective dynamics [11], the particle starting at $x = 0$ at $t = 0$ is at time t in a finite neighborhood [14] around the bottom of the bond of the renormalized lanscape at scale

$$\Gamma = T\ln t \tag{8}$$

which contains the starting point $x = 0$. After rescaling $X = \frac{x}{\Gamma^2}$, the effective dynamics gives the asymptotic exact result of the Kesten distribution [11] for the rescaled position (diffusion front). Similarly here, the effective dynamics will give the asymptotic exact result in the limit $t \to \infty$ for the rescaled variable of energy

$$w = \frac{W}{\Gamma} = \frac{W}{T\ln t} \tag{9}$$

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B. Distribution of energy for the symmetric Sinai diffusion

We introduce the probability $D_\Gamma(F, W)$ that the point $x = 0$ belongs at scale Γ to a bond of barrier F and is at potential W above the lower potential of the bond, i.e. $U(x = 0) - U_{min} = W$. It is normalized as:

$$\int_0^F dW D_\Gamma(F, W) = \frac{1}{\bar{l}_\Gamma} \int_0^\infty dl\, l P_\Gamma(F, l) \tag{10}$$

since the probability that the origin $x = 0$ belongs to a renormalized bond of barrier F and length l is $l P_\Gamma(F, l)/\bar{l}_\Gamma$. Within the effective dynamics, we may compute the probability distribution $D_t(W)$ of the energy W at time t as

$$D_t(W) = \int_{\max(\Gamma, W)}^{+\infty} dF D_\Gamma(F, W) \tag{11}$$

The consideration of the various effects of the decimations processes (see figure 1) leads to the following RG equation for $D_\Gamma(F, W)$

$$\partial_\Gamma D_\Gamma(F, W) = -2 D_\Gamma(F, W) \int_{l_2 > 0} P_\Gamma(\Gamma, l_2) \tag{12}$$

$$+ \int_{F_1 > \Gamma} \int_{F_3 > \Gamma} P_\Gamma(F_1) P_\Gamma(\Gamma) D_\Gamma(F_3, W) \delta[F - (F_1 + F_3 - \Gamma)]$$

$$+ \int_{F_1 > \Gamma} \int_{F_3 > \Gamma} \int_{0 < W' < F_1} D_\Gamma(F_1, W') P_\Gamma(\Gamma) P_\Gamma(F_3) \delta[F - (F_1 + F_3 - \Gamma)] \delta[W - (W' + F_3 - \Gamma)]$$

$$+ \int_{F_1 > \Gamma} \int_{F_3 > \Gamma} \int_{0 < W' < \Gamma} P_\Gamma(F_1) D_\Gamma(\Gamma, W') P_\Gamma(F_3) \delta[F - (F_1 + F_3 - \Gamma)] \delta[W - (W' + F_3 - \Gamma)]$$
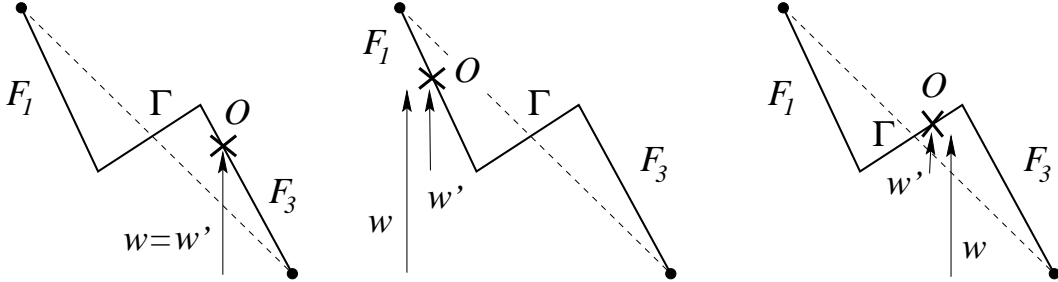

FIG. 1. Decimation of a bond of barrier Γ that gives a renormalized bond of barrier $F = F_1 + F_3 - \Gamma$: the point O can be either on the bond F_3 that doesn't change the energy $W = W'$, or on the bonds F_1 or Γ that both lead to the incrementation $W = W' + F_3 - \Gamma$ of the energy.

In the rescaled variables $\eta = \frac{F - \Gamma}{\Gamma}$ and $w = \frac{W}{\Gamma}$, the probability distribution $\mathcal{D}_\Gamma(\eta, w) = \Gamma^2 D_\Gamma(F, W)$ flows at large Γ towards the fixed point $\mathcal{D}_*(\eta, w)$ satisfying the equation

$$0 = (1 + \eta) \partial_\eta \mathcal{D}^*(\eta, w) + w \partial_w \mathcal{D}^*(\eta, w) + \int_{max(0, w-1)}^{\eta} d\eta' e^{-(\eta - \eta')} \mathcal{D}^*(\eta', w) \tag{13}$$

$$+ \int_0^{min(\eta, w)} d\eta' e^{-\eta'} \mathcal{D}^*(\eta - \eta', w - \eta') + e^{-\eta} \int_{max(0, w-1)}^{min(\eta, w)} d\eta' \mathcal{D}^*(0, w - \eta') \tag{14}$$

in the domain $\eta \in [0, +\infty)$ and $0 \leq w \leq 1 + \eta$. Note that the probability that the origin belongs at scale Γ to a bond of barrier η is already known (6)

$$\int_0^{1+\eta} dw \mathcal{D}^*(\eta, w) = \frac{1}{\bar{\lambda}} \int_0^\infty d\lambda \lambda P^*(\eta, \lambda) = \frac{1 + 2\eta}{3} e^{-\eta} \tag{15}$$

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It is now more convenient to introduce the variable $v = 1 + \eta - w$ instead to η, and to set $\mathcal{D}^*(\eta, w) = e^{-\eta}\Phi(v = 1 + \eta - w, w)$. The new function $\Phi(v, w)$ is now symmetric in (v, w) and satisfies

$$0 = v\partial_v\Phi(v, w) + w\partial_w\Phi(v, w) - (v + w)\Phi(v, w) \tag{16}$$

$$+ \int_{max(0,1-w)}^{v} dv'\Phi(v', w) + \int_{max(0,1-v)}^{w} dw'\Phi(v, w') + \int_{max(0,1-w)}^{min(1,v)} dv'\Phi(v', 1 - v') \tag{17}$$

on the domain $\{v \geq 0, w \geq 0, v + w \geq 1\}$. The function $J(v, w) = \partial_v\partial_w\Phi(v, w)$ satisfies the simpler equation

$$0 = (v\partial_v + w\partial_w + 2 - (v + w)) J(v, w) \tag{18}$$

and thus takes the form

$$J(v, w) = C(\frac{v}{w})\frac{e^{v+w}}{vw} \tag{19}$$

where $C(x)$ is an arbitrary function satisfying $C(x) = C(\frac{1}{x})$ to respect the symmetry (v, w). Since we are looking for a solution that is not exponentially growing at $(v, w) \to \infty$, it is necessay to have $C(x) \equiv 0$. The function $\Phi(v, w)$ is thus separable, and taking into account the symmetry in (v, w), we have

$$\Phi(v, w) = \phi(v) + \phi(w) \tag{20}$$

$$\tag{21}$$

where the function $\phi(v)$ has to satisfy the equation obtained by replacing the form (20) into (16)

$$0 = v\phi'(v) - (v + max(0, 1 - v))\phi(v) + \int_{max(0,1-v)}^{v} dv'\phi(v') + \int_{max(0,1-v)}^{min(1,v)} dv'\phi(v') \tag{22}$$

More explicitly, the equation becomes

$$0 = v\phi'(v) - \phi(v) + 2\int_{1-v}^{v} dv'\phi(v') \quad \text{for } 0 \leq v \leq 1 \tag{23}$$

$$0 = v\phi'(v) - v\phi(v) + \int_{0}^{v} dv'\phi(v') + \int_{0}^{1} dv'\phi(v') \quad \text{for } v \geq 1 \tag{24}$$

It is more convenient to derive these equations with respect to v

$$0 = v\phi''(v) + 2\phi(v) + 2\phi(1 - v) \quad \text{for } 0 \leq v \leq 1 \tag{25}$$
$$0 = v\phi''(v) + (1 - v)\phi'(v) \quad \text{for } v \geq 1 \tag{26}$$

and to keep the boundary condition at $v = 1$

$$0 = \phi'(1) - \phi(1) + 2\int_{0}^{1} dv'\phi(v') \tag{27}$$

In the domain $v \geq 1$, the only solution that is not exponentially growing at $v \to \infty$ is the constant solution

$$\phi(v) = \phi(1) \text{ for } v \geq 1 \tag{28}$$

and the condition at $v = 1$ becomes

$$\phi(1) = 2\int_{0}^{1} dv'\phi(v') \tag{29}$$

The constant $\phi(1)$ has to be determined through the normalization condition (15)

$$\frac{1 + 2\eta}{3} = \int_{0}^{1+\eta} dw\Phi(v = 1 + \eta - w, w) = 2\int_{0}^{1+\eta} dw\phi(w) = 2\int_{0}^{1} dw\phi(w) + 2\phi(1)\eta \tag{30}$$

which leads to

$$\phi(1) = \frac{1}{3} \tag{31}$$

$$\int_0^1 dv\phi(v) = \frac{1}{6} \tag{32}$$

We know consider the equation in the domain $v \in [0,1]$. Since the equation is not local for $\phi(v)$, it is convenient to introduce the sum

$$S(v) = \phi(v) + \phi(1-v) \tag{33}$$

that satisfies the simpler equation

$$v(1-v)S''(v) + 2S(v) = 0 \tag{34}$$

Two linearly independant solutions of this equation are

$$S^{\text{sym}}(v) = v(1-v) \tag{35}$$

$$S^{\text{antisym}}(v) = 2v(1-v)\ln\frac{v}{1-v} + 2v - 1 \tag{36}$$

Since $S(v)$ has to be symmetric in $(v, 1-v)$ by definition, and has to be normalized to

$$\int_0^1 dv S(v) = 2\int_0^1 dv\phi(v) = \frac{1}{3} \tag{37}$$

we get

$$S(v) = 2v(1-v) \tag{38}$$

and this corresponds to

$$\phi(v) = \frac{1}{3} - \frac{2}{3}(1-v)^3 \tag{39}$$

We thus finally get

$$\phi(v) = \frac{1}{3} - \frac{2}{3}(1-v)^3 \quad \text{for } 0 \le v \le 1 \tag{40}$$

$$\phi(v) = \frac{1}{3} \quad \text{for } v \ge 1 \tag{41}$$

The final result of this section is that, at large Γ, the probability $\mathcal{D}_*(\eta, w)$ that the point $x = 0$ belongs to a bond of rescaled barrier $\eta = \frac{F-\Gamma}{\Gamma}$ such that the point $x = 0$ has a rescaled potential $w = \frac{U(0)-U_{min}}{\Gamma}$ above the lower potential of the bond, reads

$$\mathcal{D}_*(\eta, w) = e^{-\eta}\Phi(1+\eta-w, w) = e^{-\eta}\left(\phi(w) + \phi(1+\eta-w)\right) \tag{42}$$

In particular, the distribution of the rescaled energy w (9) alone reads

$$\mathcal{D}_*(w) = \int_{max(0,w-1)}^\infty d\eta \mathcal{D}_*(\eta, w) \tag{43}$$

$$= \theta(w \le 1)\left(4 - 2w - 4e^{-w}\right) + \theta(w \ge 1)\left(2e - 4\right)e^{-w} \tag{44}$$

This function is represented on Figure (2). It is continuous at $w = 1$, as well as its first derivative, but the second derivative is discontinuous at $w = 1$. The contribution to the normalization of the domain $w > 1$ reads

$$\int_1^{+\infty} dw \mathcal{D}_*(w) = 2 - \frac{4}{e} = 0.52848 \tag{45}$$

The mean value is simply

$$\lim_{t \to \infty} \frac{\overline{W(t)}}{T\ln t} = \int_0^{+\infty} dw w \mathcal{D}_*(w) = \frac{4}{3} \tag{46}$$
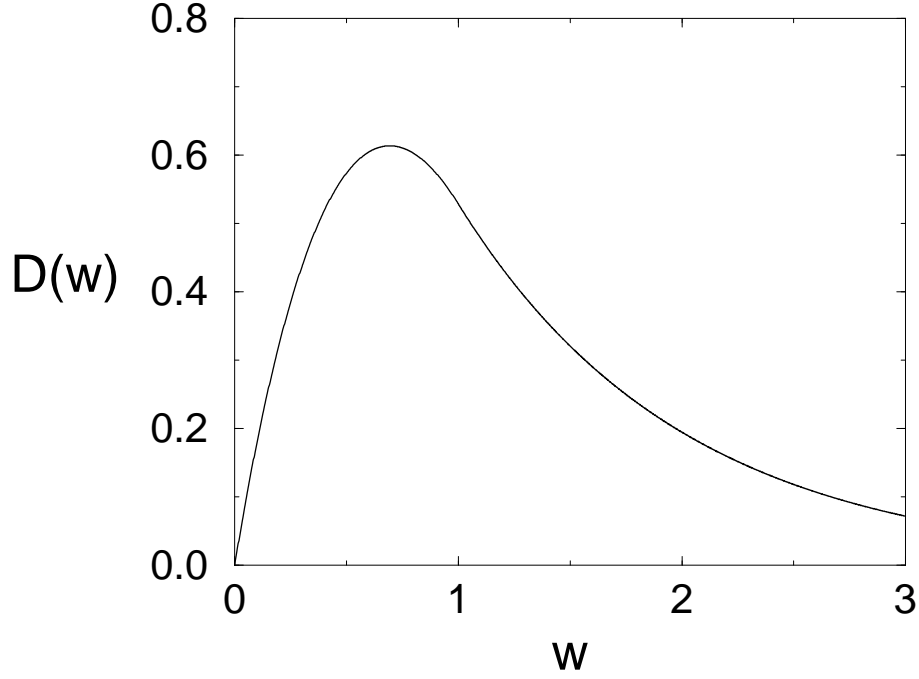
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FIG. 2. Exact asymptotic probability distribution $D_*(w)$ of the rescaled energy w.

C. Distribution of energy for the biased case

As discussed in details in Ref. [11], the RSRG method can be applied to the Sinai diffusion in the presence of a small external bias δ, and the results are valid in the double limit $\delta \to 0$ and $\Gamma \to \infty$ with the scaling variable

$$\gamma = \delta\Gamma = \delta T \ln t \tag{47}$$

being fixed.

The above study of the symmetric case is generalized to the biased case in Appendix A. The final result for the probability distribution $D_t(W)$ of $W(t)$ is the sum of the contributions of the two types of bonds:

$$D_t(W) = D_\Gamma^+(W) + D_\Gamma^-(W) \tag{48}$$

with $\Gamma = T \ln t$ and

$$D_\Gamma^\pm(W > \Gamma) = \frac{\delta^2}{\sinh^2 \delta\Gamma(\delta \coth \delta\Gamma \mp 3\delta)} \left(1 - 2e^{-\Gamma(\mp\delta+\delta \coth \delta\Gamma)}\right) e^{-(W-\Gamma)(\mp\delta+\delta \coth \delta\Gamma)} \tag{49}$$

for $W > \Gamma$, and

$$D_\Gamma^\pm(W < \Gamma) = \frac{\delta}{2\sinh^2 \delta\Gamma(\coth \delta\Gamma \mp 3)} \left(-4e^{-W(\mp\delta+\delta \coth \delta\Gamma)} + 3 \mp \coth \delta\Gamma + (1 \pm \coth \delta\Gamma)e^{\mp 2\delta W}\right) \tag{50}$$

for $0 \leq W \leq \Gamma$.

As in the symmetric case, the probability distribution $D_\Gamma^\pm(W)$ and its first derivative $\partial_W D_\Gamma^\pm(W)$ are continuous at $W = \Gamma$, but its second derivative $\partial_W^2 D_\Gamma^\pm(W)$ is discontinuous at $W = \Gamma$.

The full normalizations

$$\int_0^{+\infty} dW D_\Gamma^\pm(W) = \frac{e^{\pm 2\gamma} - 1 \mp 2\gamma}{4\sinh^2 \gamma} \tag{51}$$

correspond as it should to the ratios $\overline{l_\Gamma^\pm}/(\overline{l_\Gamma^+} + \overline{l_\Gamma^-})$ in terms of the mean lengths of the renormalized bonds.

The mean value for the $+$ case reads

$$\overline{W_+(t)} = \int_0^{+\infty} dW \, W \, D_\Gamma^{\pm}(W) \tag{52}$$

$$= \frac{e^\gamma(1 - 2e^{\gamma - \gamma \coth \gamma})(e^\gamma + \frac{\gamma}{\sinh \gamma})}{\delta(\coth \gamma - 3)} + \frac{\left(\frac{(2e^{6\gamma} - 6e^{4\gamma} + (5 - 2\gamma^2)e^{2\gamma} - 1 + 2\gamma + 4\gamma^2)}{\sinh^2 \gamma} - 16(\gamma + e^\gamma \sinh \gamma)e^{\gamma(3 - \coth \gamma)}\right)}{8\delta(e^{2\gamma} - 2)} \tag{53}$$

and varies between $\overline{W_+(t)} \to (2/3)\Gamma$ as $\delta \to 0$ and

$$\overline{W_+(t)} \underset{\gamma \to \infty}{\simeq} \frac{3}{2\delta} e^{2\gamma} \tag{54}$$

III. JOINT DISTRIBUTION OF ENERGY AND POSITION

In this section, we compute the joint distribution of position $x = x(t) - x(0)$ and energy $W = U(x(0)) - U(x(t))$ by the direct path-integral approach introduced in [15].

A. Recall of the path-integral approach

As explained in details in [15], the properties of the renormalized landscape can either be obtained as solutions of RSRG equations or by the computation of constrained path-integrals. In this second approach, the basic path integral

$$F_{[V_a, V_b]}(V_0, x_0, V, x) = \int_{V_{x_0} = V_0}^{V_l = V} DV(y) exp\left(-\int_0^x dy \left(\frac{1}{4}\left(\frac{dV}{dy}\right)^2\right)\theta_{[V_a, V_b]}^-(\{V\})\right) \tag{55}$$

over the Brownian paths going from $V(x_0) = V_0$ to $V(x) = V$ without having touched the boundaries $V = V_a$ and $V = V_b$ can be expressed in Laplace transform with respect to $(x - x_0)$ as

$$\hat{F}_{[V_a, V_b]}(V, p | V_0) = \frac{1}{W(p; V_a, V_b)} \Phi_-(V, p; V_a) \Phi_+(V_0, p; V_b) \qquad \text{if } V_a \le V \le V_0 \tag{56}$$

$$\hat{F}_{[V_a, V_b]}(V, p | V_0) = \frac{1}{W(p; V_a, V_b)} \Phi_+(V, p; V_b) \Phi_-(V_0, p; V_a) \qquad \text{if } V_0 \le V \le V_b \tag{57}$$

where Φ_- and Φ_+ are solutions of the differential equation

$$\partial_V^2 \Phi = pF \tag{58}$$

that vanish respectivly at $V = V_a$ and at $V = V_b$

$$\Phi_-(V, p; V_a) = \sinh \sqrt{p}(V - V_a) \tag{59}$$
$$\Phi_+(V, p; V_b) = \sinh \sqrt{p}(V_b - V) \tag{60}$$
$$\tag{61}$$

and where $W(p; V_a, V_b)$ is their wronskian

$$W(p; V_a, V_b) = \sqrt{p} \sinh \sqrt{p}(V_b - V_a) \tag{62}$$

As explained in [15], the probability distribution $P_\Gamma(F, l)$ corresponds to the path integral

$$B_\Gamma(F, l) = \int_{V_0 = 0}^{V_l = F} DV exp\left(-\int dx \left(\frac{1}{4}\left(\frac{dV}{dx}\right)^2\right)\Theta_\Gamma^-(\{V\})\right) \tag{63}$$

where $F \ge \Gamma$ and where $\Theta_\Gamma^-(\{V\})$ constraints the paths to remain in the interval $]0, F[$ for $0 < x < l$ and to not perform any returns of more than Γ, i.e. for two arbitrary $0 < x_1 < x_2 < l$, the potential has to satisfy $V(x_1) - V(x_2) < \Gamma$. As shown in [15], it can be obtained from the path-integral (57) as

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$$\hat{B}_\Gamma(F, p) = \frac{\sqrt{p}}{\sinh\sqrt{p}\Gamma} e^{-(F-\Gamma)\sqrt{p}\coth\sqrt{p}\Gamma} \tag{64}$$

in agreement with the fixed-point solution (6) of the RSRG equations.

Similarly, the probability $E_\Gamma(V, l, V_0)$ to go from $(V_0, 0)$ to (V, l) without making any return of more than Γ and where $V \geq V_0$ is the maximum reads in Laplace [15]

$$\hat{E}_\Gamma(V, p, V_0) = e^{-(V-V_0)\sqrt{p}\coth\sqrt{p}\Gamma} \tag{65}$$

We are now in position to compute the joint distribution of energy and position via a path-integral decomposition.

B. Joint distribution of energy and position for the symmetric case

The law is symmetric in $x \to -x$ so we will consider $x > 0$ and write the path-integral representation

$$P_t(x > 0, W) = \frac{1}{2\overline{l_\Gamma}} < \delta(V(x) - W) >_{\{V(y)\}} \tag{66}$$

$$= \frac{1}{\Gamma^2} \int_v^{+\infty} dF \int_x^{+\infty} dl \int_{V(0)=0}^{V(l)=F} \mathcal{D}V(y) e^{-\frac{1}{4}\int_0^l dy\left(\frac{dV}{dy}\right)^2} \delta(V(x) - W)\Theta_\Gamma^-\{V\} \tag{67}$$

where again $\Theta_\Gamma\{V(y)\}$ means that the path is constrained to remain in $(0, F)$ with no descending segment of more than Γ. Since this constraint doesn't factorize into the two-sides $y < x$ and $y > x$, we have to keep track of the maximum V_m reached by $V(y)$ for $0 < y < x$ to impose that the path doesn't go below $V_m - \Gamma$ for $x < y < l$.

We now compute the Laplace transform

$$\hat{P}_t(p, W) = \int_0^{+\infty} dx e^{-px} P_t(x, W) \tag{68}$$

$$\tag{69}$$

for the two cases $W > \Gamma$ and $0 < W < \Gamma$.

1. Case $W > \Gamma$

Let us introduce the following notations (see Figure 3) : We note V_m the maximum reached by $V(y)$ for $0 < y < x$, and x_m the point where it is reached $V(x_m) = V_m$. We note x_m' the first time in $x < y < l$ where $V(y)$ reaches again V_m. For the given parameters (V_m, x_m, x_m'), we have to sum over the Brownian paths satisfying the following conditions : (i) they go from $(0, 0)$ to (V_m, x_m) without making any return of more than Γ, and V_m is reached for the first time at x_m; (ii) they go from (V_m, x_m) to (W, x) without touching the absorbing boundaries at $V_m - \Gamma$ and at $V_m + \epsilon$. (iii) they go from (W, x) to (V_m, x_m') without touching the absorbing boundaries at $V_m - \Gamma$ and at $V_m + \epsilon$. (iv) they go from (V_m, x_m') to (F, l) without making any return of more than Γ and F is the maximum.

The summation over the Brownian paths satisfying all these constraints leads to

$$\hat{P}_t(x, W > \Gamma) = \frac{1}{\Gamma^2} \int_W^{W+\Gamma} dV_m \int_0^x dx_m B_\Gamma(V_m, x_m) \left(\partial_{V_0} F_{[V_m-\Gamma, V_m]}(W, x|V_0, x_m)\right)|_{V_0=V_m} \tag{70}$$

$$\int_x^{+\infty} dx_m'(\partial_{V_1} F_{[V_m-\Gamma, V_m]}(V_1, x_m'|W, x))|_{V_1=V_m} \int_{V_m}^{+\infty} dF \int_{x_m'}^{+\infty} dl E_\Gamma(F, l; V_m, x_m') \tag{71}$$

where the functions $F_{[V_a, V_b]}(V, x|V_0, x_0), B_\Gamma(F, l)$ and $E_\Gamma(F, l; V, x)$ have been defined in section (III A) and are given by the explicit expressions (57) (64) (65) in Laplace transform.

As a consequence, in Laplace transform with respect to x, we get

$$\hat{P}_t(p, W > \Gamma) = \frac{1}{\Gamma^2} \int_W^{W+\Gamma} dV_m \frac{\sqrt{p}}{\sinh\sqrt{p}\Gamma} e^{-(V_m-\Gamma)\sqrt{p}\coth\sqrt{p}\Gamma} \frac{\sinh\sqrt{p}(\Gamma - (V_m - W))}{\sinh\sqrt{p}\Gamma} \tag{72}$$

$$\frac{(\Gamma - (V_m - W))}{\Gamma} \int_{V_m}^{+\infty} dF e^{-\frac{F-V_m}{\Gamma}} \tag{73}$$

$$= \frac{\sinh\sqrt{p}\Gamma}{\Gamma^2\sqrt{p}} e^{-(W-\Gamma)\sqrt{p}\coth\sqrt{p}\Gamma} \left(1 - 2\cosh\sqrt{p}\Gamma e^{-\Gamma\sqrt{p}\coth\sqrt{p}\Gamma}\right) \tag{74}$$
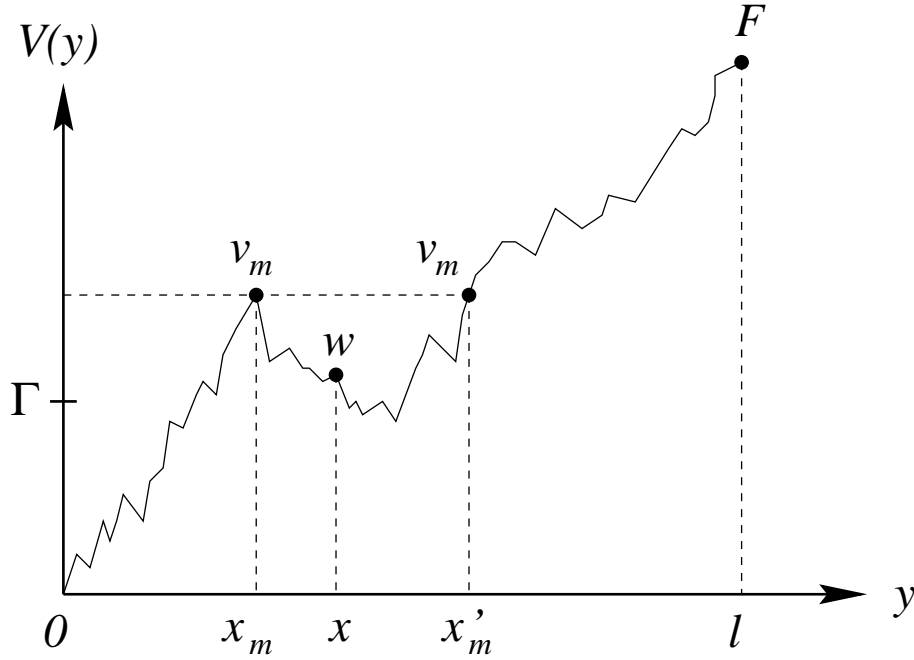
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FIG. 3. Properties of the renormalized bond for $W > \Gamma$.

Rescaling the variables

$$\mathcal{P}(X = \frac{x}{\Gamma^2}, w = \frac{W}{\Gamma}) = \Gamma^3 P_t(x, W) \tag{75}$$

we get the Laplace transform of the exact asymptotic distribution of the rescaled variables X and w

$$\hat{\mathcal{P}}(s, w > 1) \equiv \int_0^{+\infty} dX e^{-sX} \mathcal{P}(X, w > 1) \tag{76}$$

$$= \frac{\sinh\sqrt{s}}{\sqrt{s}} \left(e^{\sqrt{s}\coth\sqrt{s}} - 2\cosh\sqrt{s} \right) e^{-w\sqrt{s}\coth\sqrt{s}} \tag{77}$$

In particular, the distribution of rescaled energy w alone reads (by taking into account the two sides $X > 0$ and $X < 0$)

$$\mathcal{P}(w > 1) = 2\hat{\mathcal{P}}(s = 0, w > 1) = 2(e - 2)e^{-w} \tag{78}$$

in agrement with the RSRG result (44).

2. *Case* $W < \Gamma$

For $W < \Gamma$, there are two contributions coming from the cases $V_m > \Gamma$ and $V_m < \Gamma$, where V_m is again the maximum reached by the potential $V(y)$ for $0 < y < x$

$$P_t(x, W < \Gamma) = P_t^{(1)}(x, W < \Gamma) + P_t^{(2)}(x, W < \Gamma) \tag{79}$$

For $V_m > \Gamma$, we have the same decomposition as in (71), except that here we have to integrate over $V_m \in (\Gamma, W + \Gamma)$ (instead of $V_m \in (W, W + \Gamma)$)

$$P_t^{(1)}(x, W) = \frac{1}{\Gamma^2} \int_\Gamma^{W+\Gamma} dV_m \int_0^x dx_m B_\Gamma(V_m, x_m) \left(\partial_{V_0} F_{[V_m - \Gamma, V_m]}(W, x | V_0, x_m) \right) |_{V_0 = V_m} \tag{80}$$

$$\int_x^{+\infty} dx'_m (\partial_{V_1} F_{[V_m - \Gamma, V_m]}(V_1, x'_m | W, x) |_{V_1 = V_m} \int_{V_m}^{+\infty} dF \int_{x'_m}^{+\infty} dl E_\Gamma(F, l; V_m, x'_m) \tag{81}$$

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In Laplace transform, we thus get

$$P^{(1)}(p, W < \Gamma) = \frac{1}{\Gamma^2} \int_{\Gamma}^{W+\Gamma} dV_m \frac{\sqrt{p}}{\sinh \sqrt{p}\Gamma} e^{-(V_m - \Gamma)\sqrt{p}\coth \sqrt{p}\Gamma} \frac{\sinh \sqrt{p}(\Gamma - (V_m - W))}{\sinh \sqrt{p}\Gamma} \tag{82}$$

$$\frac{(\Gamma - (V_m - W))}{\Gamma} \int_{V_m}^{+\infty} dF e^{-\frac{F - V_m}{\Gamma}} \tag{83}$$

$$= \frac{1}{\Gamma^2} \left(\frac{\sinh \sqrt{p}(2\Gamma - W)}{\sqrt{p}} - W \frac{\sinh \sqrt{p}(\Gamma - W)}{\sinh \sqrt{p}\Gamma} - \frac{\sinh 2\sqrt{p}\Gamma}{\sqrt{p}} e^{-W\sqrt{p}\coth \sqrt{p}\Gamma} \right) \tag{84}$$

Rescaling the variables as in (75), the first contribution in (79) reads

$$\hat{\mathcal{P}}^{(1)}(s, w < 1) = \int_0^{+\infty} dX e^{-sX} \mathcal{P}^{(1)}(X, w < 1) \tag{85}$$

$$= \frac{\sinh \sqrt{s}(2 - w)}{\sqrt{s}} - w \frac{\sinh \sqrt{s}(1 - w)}{\sinh \sqrt{s}} - \frac{\sinh 2\sqrt{s}}{\sqrt{s}} e^{-w\sqrt{s}\coth \sqrt{s}} \tag{86}$$

We now consider the second contribution corresponding to the cases where the maximum V_m reached by $V(y)$ for $0 < y < x$ satisfies $V_m < \Gamma$ (see Figure 4). We then have to sum over the Brownian paths satisfying the following conditions : (i) they go from $(0,0)$ to (W, x) without touching the bondaries at $V = 0$ and $V = \Gamma$. (ii) they go from (W, x) to (Γ, z) without touching the absorbing boundaries at $V = 0$ and at $\gamma + \epsilon$. (iv) they go from (Γ, z) to (F, l) without making any return of more than Γ and F is the maximum.
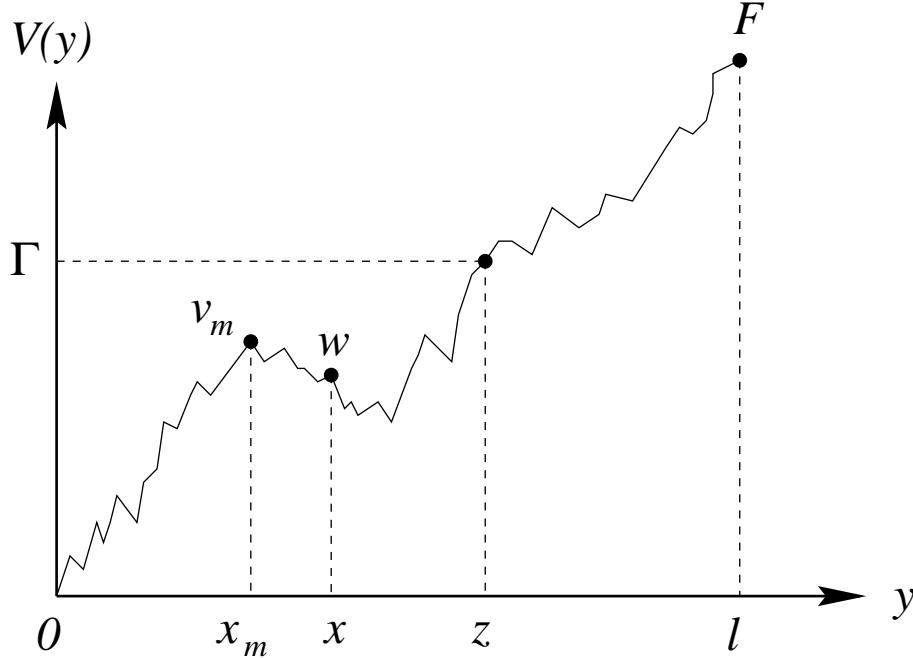

FIG. 4. Properties of the renormalized bond for $V_m < \Gamma$.

The summation over the Brownian paths satisfying all these constraints leads to

$$\hat{P}_t^{(2)}(x, W) = \frac{1}{\Gamma^2} \left(\partial_{V_0} F_{[0,\Gamma]}(V_0, 0; W, x) \right)_{V_0 = 0} \int_x^\infty dz \left(-\partial_{V'} F_{[0,\Gamma]}(W, x; , V', z) \right)_{V' = \Gamma} \tag{87}$$

$$\int_{\Gamma}^{+\infty} dF \int_z^{+\infty} dl E_\Gamma(\Gamma, z; F, l) \tag{88}$$

and thus in Laplace with respect to x, we get

$$P^{(2)}(p, W) = \frac{1}{\Gamma^2} \left(\partial_{V_0} F_{[0,\Gamma]}(V_0, W; p) \right)_{V_0 = 0} \left(-\partial_{V'} F_{[0,\Gamma]}(W, V'; 0) \right)_{V' = \Gamma} \int_{\Gamma}^{+\infty} dF E_\Gamma(\Gamma, F; 0) \tag{89}$$

$$= W \frac{\sinh(\Gamma - W)\sqrt{p}}{\Gamma^2 \sinh \Gamma \sqrt{p}} \tag{90}$$

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Rescaling the variables as in (75), the second contribution in (79) reads

$$\hat{\mathcal{P}}^{(2)}(s, w < 1) \equiv \int_0^{+\infty} dX e^{-sX} \mathcal{P}^{(1)}(X, w < 1) = w \frac{\sinh(1 - w)\sqrt{s}}{\sinh \sqrt{s}} \tag{91}$$

Adding the two contributions (86) and (91), we finally get the exact asymptotic distribution of the joined rescaled variables (X, w) in the domain $w < 1$

$$\hat{\mathcal{P}}(s, w < 1) = \hat{\mathcal{P}}^{(1)}(s, w < 1) + \hat{\mathcal{P}}^{(2)}(s, w < 1) \tag{92}$$

$$= \frac{\sinh \sqrt{s}(2 - w)}{\sqrt{s}} - \frac{\sinh 2\sqrt{s}}{\sqrt{s}} e^{-w\sqrt{s} \coth \sqrt{s}} \tag{93}$$

In particular, the distribution of rescaled energy alone w reads

$$\mathcal{P}(w < 1) = 2\hat{\mathcal{P}}(s = 0, w < 1) \tag{94}$$

$$= 2(2 - w - 2e^{-w}) \tag{95}$$

in agrement with the RSRG solution (44).

3. Kesten distribution for the rescaled position alone

A check of the above result is that one recovers the Kesten distribution, obtained in [11] via the RSRG, upon integration over the energy.

For the domain $w > 1$, the distribution of the rescaled position X has for Laplace transform (86)

$$\hat{\mathcal{P}}_>(s) = \int_1^{+\infty} dw \hat{\mathcal{P}}(s, w > 1) \tag{96}$$

$$= \frac{\sinh^2 \sqrt{s}}{s} \left(\frac{1}{\cosh \sqrt{s}} - 2e^{-\sqrt{s} \coth \sqrt{s}} \right) \tag{97}$$

whereas it is given in the domain $w < 1$ by (91)

$$\hat{\mathcal{P}}_<(s) = \int_0^1 dw \hat{\mathcal{P}}(s, w < 1) \tag{98}$$

$$= \frac{1}{s} \left(1 - \cosh \sqrt{s} + 2 \sinh^2 \sqrt{s} e^{-\sqrt{s} \coth \sqrt{s}} \right) \tag{99}$$

Adding these two contributions, we recover as it should the Kesten distribution [11]

$$\hat{\mathcal{P}}(s) = \hat{\mathcal{P}}_<(s) + \hat{\mathcal{P}}_>(s) = \frac{1}{s} \left(1 - \frac{1}{\cosh \sqrt{s}} \right) \tag{100}$$

4. Displacements for fixed energy

It is interesting to use the above solution to compute the moments of the distance traveled by the particle given that the (rescaled) potential energy lost is a fixed $w = W/T \ln t$. The probability distribution of the rescaled distance X for a given rescaled energy $q_w(X) = \mathcal{P}(X, w)/\int dX \mathcal{P}(X, w)$ has for Laplace transform (77,93)

$$q_w(s) = \frac{\hat{\mathcal{P}}(s, w > 1)}{\hat{\mathcal{P}}(0, w > 1)} = \frac{\sinh \sqrt{s}}{\sqrt{s}(e - 2)} \left(e^{\sqrt{s} \coth \sqrt{s}} - 2 \cosh \sqrt{s} \right) e^{w(1 - \sqrt{s} \coth \sqrt{s})} \quad \text{for} \quad w > 1 \tag{101}$$

$$q_w(s) = \frac{\hat{\mathcal{P}}(s, w < 1)}{\hat{\mathcal{P}}(0, w < 1)} = \frac{\frac{\sinh \sqrt{s}(2 - w)}{\sqrt{s}} - \frac{\sinh 2\sqrt{s}}{\sqrt{s}} e^{-w\sqrt{s} \coth \sqrt{s}}}{(2 - w) - 2e^{-w}} \quad \text{for} \quad w < 1 \tag{102}$$

The moments

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$$c_n(w) = \int_0^{+\infty} dX\, X^n q_w(X) = < \left(\frac{|x(t) - x(0)|}{(T \ln t)^2}\right)^n >_w \tag{103}$$

represent the moments of the distance traveled over the set of trajectories and environments such that the potential energy lost is w.

In particular, the first moment reads

$$c_1(w) = \frac{w}{3} + \frac{8 - 3e}{6(e - 2)} \qquad \text{for} \quad w > 1 \tag{104}$$

$$c_1(w) = \frac{(2 - w)(4 - e^w(2 - w)^2)}{6(e^w(2 - w) - 2)} \quad \text{for} \qquad w < 1 \tag{105}$$

and order n similar polynomial extensions are obtained for the n-th moment. As a comparison, note that c_1 is $5/12$ for the full Kesten diffusion front.

These moments $c_n(w)$ are increasing function of w as expected since roughly speaking larger w correspond to larger displacements. For $w = 0$, they simply vanish $c_n(w) = 0$, which correspond to

$$q_{w=0}(s) = 1 \quad \text{i.e.} \qquad q_{w=0}(X) = \delta(X) \tag{106}$$

For large w, the asymptotic behavior of the moments is simply given by

$$c_n(w) \underset{w \to \infty}{\simeq} \left(\frac{w}{3}\right)^n \tag{107}$$

This interesting property can be understood by considering the large w limit for the distribution (101), which is dominated by the region $s \to 0$:

$$q_w(s) \underset{w \to \infty}{\simeq} e^{-w\frac{s}{3}} \tag{108}$$

corresponding after Laplace inversion to

$$q_w(X) \underset{w \to \infty}{\simeq} \delta(X - \frac{w}{3}) \tag{109}$$

i.e as w becomes large, in terms of the original variables, we have

$$\frac{x(t)}{(T \ln t)^2} = \frac{1}{3}\frac{W(t)}{T \ln t} \tag{110}$$

with probability one.

C. Joint distribution of energy and position for the biased case

The above calculation can be generalized in the presence of a small bias.

The joint distribution of energy and position is given for the biased case by the following path-integrals with $x > 0$

$$P_t(\pm x, W) = \frac{1}{l_+ + l_-} < \delta(V_\pm(x) - W) >_{\{V_\pm(y)\}} \tag{111}$$

$$= \frac{\delta^2}{\sinh^2 \delta\Gamma} \int_W^{+\infty} dF \int_x^{+\infty} dl \int_{V(0)=0}^{V(l)=F} \mathcal{D}V(y) e^{-\frac{1}{4}\int_0^l dy\left(\frac{dV}{dy} \mp 2\delta\right)^2} \Theta_\Gamma\{V(y)\}\delta(V(x) - W) \tag{112}$$

The explicit computation is done in Appendix (A 3) and leads to the following final results in Laplace transform for $W > \Gamma$ and $W < \Gamma$ respectivly

$$\hat{P}_t^\pm(p, W > \Gamma) = \frac{\delta^2}{\sinh^2 \delta\Gamma} \frac{\frac{\sqrt{p+\delta^2}}{\sinh\sqrt{p+\delta^2}\Gamma}e^{\mp\delta\Gamma} - 2(\mp\delta + \delta\coth\delta\Gamma)e^{-\Gamma(\mp\delta+\delta\coth\delta\Gamma)}}{\left(\frac{\delta^2+p}{\sinh^2\sqrt{p+\delta^2}\Gamma} + 4\delta^2 \mp 4\delta\sqrt{p+\delta^2}\coth\sqrt{p+\delta^2}\Gamma\right)} e^{-(W-\Gamma)(\mp\delta+\sqrt{p+\delta^2}\coth\sqrt{p+\delta^2}\Gamma)} \tag{113}$$

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$$\hat{P}_t^{\pm}(p, W < \Gamma) = \frac{\delta^2}{\sinh^2 \delta\Gamma \left(\frac{p+\delta^2}{\sinh^2 \sqrt{p+\delta^2}\Gamma} \mp 4\delta(\mp\delta + \sqrt{p+\delta^2}\coth\sqrt{p+\delta^2}\Gamma) \right)} \tag{114}$$

$$[-2(\mp\delta + \sqrt{p+\delta^2}\coth\sqrt{p+\delta^2}\Gamma)e^{-W(\mp\delta+\sqrt{p+\delta^2}\coth\sqrt{p+\delta^2}\Gamma)} \tag{115}$$

$$+2(\mp\delta + \sqrt{p+\delta^2}\coth\sqrt{p+\delta^2}\Gamma)\cosh\sqrt{p+\delta^2}W\cosh\delta W \tag{116}$$

$$-(\sqrt{p+\delta^2}(1+\coth^2\sqrt{p+\delta^2}\Gamma) \mp 2\delta\coth\sqrt{p+\delta^2}\Gamma))\sinh\sqrt{p+\delta^2}W\cosh\delta W \tag{117}$$

$$+\frac{1}{\delta}(p - (\mp\delta + \sqrt{p+\delta^2}\coth\sqrt{p+\delta^2}\Gamma)^2)\cosh\sqrt{p+\delta^2}W\sinh\delta W \tag{118}$$

$$+(\mp\delta + \sqrt{p+\delta^2}\coth\sqrt{p+\delta^2}\Gamma)(-2\coth\sqrt{p+\delta^2}\Gamma + \frac{\sqrt{p+\delta^2}}{\delta\sinh^2\sqrt{p+\delta^2}\Gamma})\sinh\sqrt{p+\delta^2}W\sinh\delta W] \tag{119}$$

$$+\frac{\delta\sinh\delta W\sinh(\Gamma-W)\sqrt{p+\delta^2}}{\sinh^2\delta\Gamma\sinh\Gamma\sqrt{p+\delta^2}} \tag{120}$$

For the special case $p = 0$, these expressions coincide as it should with the results (49) and (50) corresponding to the distribution of the energy W alone.

On the other hand, the integration over the energy W gives

$$\hat{P}_t^{>\pm}(p) = \int_{\Gamma}^{+\infty} dW \hat{P}_t^{\pm}(p, W > \Gamma) \tag{121}$$

$$= \frac{\delta^2}{\sinh^2\delta\Gamma} \frac{\frac{\sqrt{p+\delta^2}}{\sinh\sqrt{p+\delta^2}\Gamma}e^{\mp\delta\Gamma} - 2(\mp\delta + \delta\coth\delta\Gamma)e^{-\Gamma(\mp\delta+\delta\coth\delta\Gamma)}}{(\frac{\delta^2+p}{\sinh^2\sqrt{p+\delta^2}\Gamma} + 4\delta^2 \mp 4\delta\sqrt{p+\delta^2}\coth\sqrt{p+\delta^2}\Gamma)(\mp\delta + \sqrt{p+\delta^2}\coth\sqrt{p+\delta^2}\Gamma)} \tag{122}$$

and

$$\hat{P}_t^{<\pm}(p) = \int_0^{+\Gamma} dW \hat{P}_t^{\pm}(p, W < \Gamma) \tag{123}$$

$$= \frac{\delta^2}{p\sinh^2\delta\Gamma((\sqrt{p+\delta^2}\coth\sqrt{p+\delta^2}\Gamma - 2\delta)^2 - p - \delta^2))}[2pe^{-\Gamma(\mp\delta+\sqrt{p+\delta^2}\coth\sqrt{p+\delta^2}\Gamma)} \tag{124}$$

$$+\frac{\sqrt{p+\delta^2}\sinh\delta}{\delta\sinh\sqrt{p+\delta^2}\Gamma}(\delta^2 + 3\delta^2\coth\delta\Gamma - \delta\sqrt{p+\delta^2}\coth\sqrt{p+\delta^2}\Gamma(3+\coth\delta\Gamma) + \frac{p+\delta^2}{\sinh^2\sqrt{p+\delta^2}\Gamma})] \tag{125}$$

$$+\frac{\delta^2}{p\sinh^2\delta\Gamma}\left(1 - \frac{\sqrt{p+\delta^2}\sinh\delta\Gamma}{\delta\sinh\sqrt{p+\delta^2}\Gamma}\right) \tag{126}$$

whose sum coincides as it should with the distribution of the position alone [11]

$$\hat{P}_t^{\pm}(p) = \hat{P}_t^{>\pm}(p) + \hat{P}_t^{<\pm}(p) \tag{127}$$

$$= \frac{\delta^2}{p\sinh^2\delta\Gamma}\left(1 - \frac{\frac{\sqrt{p+\delta^2}}{\sinh\sqrt{p+\delta^2}\Gamma}e^{\mp\delta\Gamma}}{(\pm\delta + \sqrt{p+\delta^2}\coth\sqrt{p+\delta^2}\Gamma)}\right) \tag{128}$$

IV. RESULTS IN THE PRESENCE OF A REFLECTING BOUNDARY

In this section, we consider the case where there is a reflecting boundary at the starting point $x = 0$ [11], with possibly some bias. It is interesting as it can be studied easily using the RSRG.

A. Joint distribution of energy and position on the half-line

In the case of a bias, the joint probability distribution of $E_\Gamma^\pm(W,l)$ of the energy-loss $W = U(t=0) - U(t)$ and the distance $l = x(t) - x(0)$ satisfies the following RG equation

$$\partial_\Gamma E_\Gamma^\pm(W,l) = -E_\Gamma^\pm(W,l) \int_0^\infty dl' P_\Gamma^\mp(\Gamma,l') \tag{129}$$

$$+ \int_0^W dW_1 \int_0^\infty dl_1 \int_0^\infty dl_2 \int_\Gamma^\infty dF_3 \int_0^\infty dl_3 E_\Gamma^\pm(W_1,l_1) P_\Gamma^\mp(\Gamma,l_2) P_\Gamma^\pm(F_3,l_3) \tag{130}$$

$$\delta(W - (W_1 + F_3 - \Gamma))\delta(l - (l_1 + l_2 + l_3)) \tag{131}$$

For large Γ using the properties of the fixed point solution (A1) for $P^\pm$, and the properties (A4) of the functions U and u, this can be rewritten in the Laplace variable with respect to the length as

$$\partial_\Gamma E_\Gamma^\pm(W,p) = -u_\Gamma^\mp(0)E_\Gamma^\pm(W,p) + U_\Gamma^+(p)U_\Gamma^-(p) \int_0^W dW_1 E_\Gamma^\pm(W_1,p)e^{-(W-W_1)u_\Gamma^\pm(p)} \tag{132}$$

and we finally get that the joint distribution of position l and energy W takes the very simple form

$$E_\Gamma^\pm(W,p) = u_\Gamma^\pm(0)e^{-Wu_\Gamma^\pm(p)} \tag{133}$$

The distribution of the energy W alone is a simple exponential

$$E_\Gamma^\pm(W) = u_\Gamma^\pm(0)e^{-Wu_\Gamma^\pm(0)} \tag{134}$$

contrary to the case studied in previous sections where the diffusion takes place on the full line. The absence of nonanalytic behaviour is traced to the fact that here decimations of the bond near the boundary do not occur.

B. Aging for the joint distribution on the half-line

It is now possible to obtain the correlation of the energy at two different times, i.e. the aging of the system in energy.

In the case of a reflecting boundary at the origin, the single time diffusion front is simply

$$\overline{P(W',x',t'|0,0,0)} = E_{\Gamma'}^+(W',x') \tag{135}$$

whose Laplace transform is given in equation (133). Similarly, the two-time diffusion front $\overline{P(W,x,t,W',x',t'|00)} = E_{\Gamma,\Gamma'}^+(W,x,W',x')$, satisfies the RG equation given in (131) but with the initial condition at $\Gamma = \Gamma'$:

$$E_{\Gamma=\Gamma',\Gamma'}^+(W,x,W',x') = \delta(W-W')\delta(x-x')E_{\Gamma'}^+(W',x') \tag{136}$$

Within the effective dynamics, this system has the flavor of a directed model, since $W - W'$ and $x(t) - x(t')$ are always positive. It is thus convenient to define the Laplace transforms

$$\hat{E}_{\Gamma,\Gamma'}^+(\lambda,p,W',p') = \int_0^{+\infty} dx'e^{-p'x'} \int_{x'}^\infty dxe^{-p(x-x')} \int_{W'}^\infty dWe^{-\lambda(W-W')} E_{\Gamma,\Gamma'}^+(W,x,W',x') \tag{137}$$

Using the fixed point solution (A1) for $P_\Gamma^\pm$ together with the properties (A4), the above RG equation simplifies into

$$\partial_\Gamma \ln \hat{E}_{\Gamma,\Gamma'}^+(\lambda,p,W',p') = -u_\Gamma^-(0) + \frac{U_\Gamma^+(p)U_\Gamma^-(p)}{\lambda + u_\Gamma^+(p)} = \partial_\Gamma \ln \left(\frac{u_\Gamma^+(0)}{\lambda + u_\Gamma^+(p)} \right) \tag{138}$$

with the initial condition

$$\hat{E}_{\Gamma=\Gamma',\Gamma'}^+(\lambda,p,W',p') = E_{\Gamma'}^+(W',p') = u_{\Gamma'}^+(0)e^{-W'u_{\Gamma'}^+(p')} \tag{139}$$

We thus obtain

$$\hat{E}_{\Gamma,\Gamma'}^+(\lambda, p, W', p') = E_{\Gamma'}^+(W', p') \frac{u_\Gamma^+(0)}{u_{\Gamma'}^+(0)} \frac{\lambda + u_{\Gamma'}^+(p)}{\lambda + u_\Gamma^+(p)} \tag{140}$$

or more expliciltly in terms of (W, W') and (p, p')

$$\hat{E}_{\Gamma,\Gamma'}^+(W, p, W', p') = u_\Gamma^+(0)e^{-W'u_{\Gamma'}^+(p')} \left(\delta(W - W') + (u_{\Gamma'}^+(p) - u_\Gamma^+(p))e^{-(W-W')u_\Gamma^+(p)} \right) \tag{141}$$

In particular for $p = 0 = p'$, we obtain the aging properties of the energy alone

$$E_{\Gamma,\Gamma'}^+(W, W') = u_\Gamma^+(0)e^{-W'u_{\Gamma'}^+(0)} \left(\delta(W - W') + (u_{\Gamma'}^+(0) - u_\Gamma^+(0))e^{-(W-W')u_\Gamma^+(0)} \right) \tag{142}$$

For the symmetric case ($\delta = 0$), it simplifies into

$$E_{\Gamma,\Gamma'}(W, W') = \frac{1}{\Gamma}e^{-W'/\Gamma'} \left(\delta(W - W') + (\frac{1}{\Gamma'} - \frac{1}{\Gamma})e^{-(W-W')/\Gamma} \right) \tag{143}$$

In particular, the two-time Energy Correlation reads

$$\overline{U(t)U(t')} = \int_0^{+\infty} dW' \int_{W'}^{+\infty} dW \ WW' E_{\Gamma,\Gamma'}^+(W, W') = \frac{1}{u_{\Gamma'}^+(0)} \left(\frac{1}{u_{\Gamma'}^+(0)} + \frac{1}{u_\Gamma^+(0)} \right) \tag{144}$$

In the limit $\gamma = \Gamma\delta = T\delta \ln t \gg 1$ $\gamma' = \Gamma'\delta = T\delta \ln t' \gg 1$, the influence of the wall vanishes in that limit, and the result should coincide with the result of the full model without a wall.

V. DISTRIBUTION OF THE MAGNETIZATION IN THE 1D SPIN GLASS

Finally, let us indicate how the above results can be translated in the language of the one dimensional Ising spin glass in an external field. We refer the reader to [16] for the details of the study of the spin glass by the RSRG method. We consider the Hamiltonian

$$H = -\sum_{i=1}^{i=N-1} J_i \sigma_i \sigma_{i+1} - \sum_{i=1}^{i=N} h\sigma_i \tag{145}$$

with a bimodal distribution of random bonds $J_i = J\epsilon_i$. In the absence of a field $h = 0$ the spin glass has two ground states $\pm\sigma_i^0$ with $\sigma_i^0 = \epsilon_1..\epsilon_{i-1}$. In the presence of a field h the ground state of the spin glass is made of domains of either zero field ground states. At finite time t after the quench, these domain grow until they reach their equilibrium size at equilibration time $\Gamma_{eq} = T \ln t_{eq} = 4J$. Each of these domain consists of an interval between two frustrated bonds (i.e. such that $J_i\sigma_i\sigma_{i+1} < 0$). During the coarsening process the number of such bonds decrease with time.

Choosing a point O randomly in space one can thus, at time t, corresponding to the RSRG scale $\Gamma = T \ln t$ denote by $i = R$ (and $i = L$ respectively) the closest frustrated bond to the right (respectively the left) of point 0. In the spin glass the (absolute value of the) magnetization plays the role of the energy barriers in the Sinai model, more precisely:

$$|M_{L0}| \equiv |\sum_{L \le i \le O} \sigma_i| = |\sum_{L \le i \le O} \sigma_i^0| = \frac{1}{2h}|V(L) - V(0)| \tag{146}$$

where V is a Sinai type potential. It is then easy to see that if one defines

$$w_1 = \frac{|M_{L0}(t)|}{T \ln t} \qquad w_2 = \frac{|M_{0R}(t)|}{T \ln t} \tag{147}$$

then the joint distribution $P(w_1, w_2)$ of the scaling variables w_1, w_2 has been determined in Section II:

$$P(w_1, w_2) = \mathcal{D}_*(\eta = w_1 + w_2 - 1, w_1) = e^{-w_1 - w_2 + 1}(\phi(w_1) + \phi(w_2))\theta(w_1 + w_2 - 1) \tag{148}$$

where $\phi(w)$ is given in (41). It exhibits, in particular, the same nonanaliticity as the distribution of the potential energy loss of a particle in the Sinai model. The distribution of the total magnetization $|M_{L0}(t) + M_{0R}(t)| = |M_{L0}(t)| + |M_{0R}(t)|$ is just the barrier distribution, i.e. a simple exponential, as found in [16]. Note also, that the joint distribution of the *equilibrium* magnetizations is also (148) with the definitions (147) replacing $T \ln t \to 4J$.

APPENDIX A: GENERALIZATION TO THE BIASED CASE

1. Recall of main results of the real space renormalization method

In the presence of an external drift, there are two probability distributions $P_\Gamma^\pm(F, l)$ describing respectivly the ascending and the descending bonds. The asymptotic distributions read [11]

$$P_\Gamma^\pm(F, p) = U_\Gamma^\pm(p)e^{-(F-\Gamma)u_\Gamma^\pm(p)} \tag{A1}$$

$$u_\Gamma^\pm(p) = \sqrt{p + \delta^2} \coth\left[\Gamma\sqrt{p + \delta^2}\right] \mp \delta \tag{A2}$$

$$U_\Gamma^\pm(p) = \frac{\sqrt{p + \delta^2}}{\sinh\left[\Gamma\sqrt{p + \delta^2}\right]}e^{\mp\delta\Gamma} \tag{A3}$$

It is also usefull to recall the evolution equations

$$\partial_\Gamma u_\Gamma^\pm(p) = -U_\Gamma^+(p)U_\Gamma^-(p) \tag{A4}$$
$$\partial_\Gamma U_\Gamma^\pm(p) = -u_\Gamma^\mp(p)U_\Gamma^\pm(p) \tag{A5}$$

In particular, the distributions of the barriers alone are in terms of the variable $z = F - \Gamma$

$$P_\Gamma^\pm(z) = u_\Gamma^\pm(0)e^{-zu_\Gamma^\pm(0)} \tag{A6}$$

$$u_\Gamma^\pm(0) = \frac{\delta}{\sinh(\Gamma\delta)}e^{\mp\delta\Gamma} \tag{A7}$$

Finally, it is also convenient to introduce the density of renormalized valleys

$$n_\Gamma = \frac{1}{\overline{l_\Gamma^+} + \overline{l_\Gamma^-}} = \frac{\delta^2}{\sinh^2(\Gamma\delta)} = u_\Gamma^+(0)u_\Gamma^-(0) \tag{A8}$$

2. Distribution of energy via the RSRG equation

We introduce the probability $D_\Gamma^\pm(F, W)$ that the point $x = 0$ belongs at scale Γ to a $\pm$ bond of barrier F and is at potential W above the lower potential of the bond (i.e. $U(x = 0) - U_{min} = W$). They are normalized to

$$\int_0^F dW D_\Gamma^\pm(F, W) = \frac{1}{\overline{l_\Gamma^+} + \overline{l_\Gamma^-}} \int_0^\infty dll P_\Gamma^\pm(F, l) \tag{A9}$$

The RG equations for $D_\Gamma^\pm(F, W)$ are

$$\partial_\Gamma D_\Gamma^\pm(F, W) = -2D_\Gamma^\pm(F, W)P_\Gamma^\mp(\Gamma) \tag{A10}$$

$$+ \int_\Gamma^\infty dF_1 \int_\Gamma^\infty dF_3 P_\Gamma^\pm(F_1)P_\Gamma^\mp(\Gamma)D_\Gamma^\pm(F_3, W)\delta[F - (F_1 + F_3 - \Gamma)] \tag{A11}$$

$$+ \int_\Gamma^\infty dF_1 \int_\Gamma^\infty dF_3 \int_0^{F_1} dW' D_\Gamma^\pm(F_1, W')P_\Gamma^\mp(\Gamma)P_\Gamma^\pm(F_3)\delta[F - (F_1 + F_3 - \Gamma)]\delta[W - (W' + F_3 - \Gamma)] \tag{A12}$$

$$+ \int_\Gamma^\infty dF_1 \int_\Gamma^\infty dF_3 \int_0^\Gamma dW' P_\Gamma^\pm(F_1)E_\Gamma^\mp(\Gamma, W')P_\Gamma^\pm(F_3)\delta[F - (F_1 + F_3 - \Gamma)]\delta[W - (W' + F_3 - \Gamma)] \tag{A13}$$

Using the variable $z = F - \Gamma$ and the fixed point solutions $P_\Gamma^\pm(z)$ (A7), the RG equations become

$$(\partial_\Gamma - \partial_z)D_\Gamma^\pm(z, W) = -2u_\Gamma^\mp D_\Gamma^\pm(z, W) + u_\Gamma^- u_\Gamma^+ \int_{max(0, W-\Gamma)}^z dz' e^{-(z-z')u_\Gamma^\pm}D_\Gamma^\pm(z', W) \tag{A14}$$

$$+ u_\Gamma^- u_\Gamma^+ \int_0^{min(z, W)} dz' e^{-z'u_\Gamma^\pm}D_\Gamma^\pm(z - z', W - z') + (u_\Gamma^\pm)^2 e^{-zu_\Gamma^\pm} \int_{max(0, W-\Gamma)}^{min(z, W)} dz' D_\Gamma^\mp(0, W - z') \tag{A15}$$

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Using (A8) it is convenient to set

$$D_\Gamma^\pm(z, W) = n_\Gamma u_\Gamma^\pm e^{-zu_\Gamma^\pm} \Psi_\Gamma^\pm(V = z + \Gamma - W, W) \tag{A16}$$

and to write the RG equations for $\Psi_\Gamma^\pm(V, W)$

$$[\partial_\Gamma + (V + W - \Gamma)n_\Gamma] \Psi_\Gamma^\pm(V, W) \tag{A17}$$

$$= n_\Gamma \left(\int_{max(0,\Gamma-W)}^V dV' \Psi_\Gamma^\pm(V', W) + \int_{max(0,\Gamma-V)}^W dW' \Psi_\Gamma^\pm(V, W') + \int_{max(0,\Gamma-W)}^{min(\Gamma,V)} dV' \Psi_\Gamma^\mp(V', \Gamma - V') \right) \tag{A18}$$

Applying $\partial_V \partial_W$ to these equations lead to the simple equations

$$[\partial_\Gamma + (V + W - \Gamma)n_\Gamma] \partial_V \partial_W \Psi_\Gamma^\pm(V, W) = 0 \tag{A19}$$

Since we are loking for solutions that are not exponentially growing as $(V, W) \to +\infty$, we are led, as in the symmetric case, to

$$\partial_V \partial_W \Psi_\Gamma^\pm(V, W) = 0 \tag{A20}$$

It is now natural to set

$$\Psi_\Gamma^+(V, W) = \Psi_\Gamma^-(V, W) = \Psi_\Gamma(V, W) = \psi_\Gamma(V) + \psi_\Gamma(W) \tag{A21}$$

and the equation for $\psi_\Gamma(V)$ reads

$$[\partial_\Gamma + (V - \Gamma + max(0, \Gamma - V))n_\Gamma] \psi_\Gamma(V) \tag{A22}$$

$$= +n_\Gamma \left(\int_{max(0,\Gamma-V)}^V dV' \psi_\Gamma(V') + \int_{max(0,\Gamma-V)}^{min(\Gamma,V)} dV' \psi_\Gamma(V') \right) \tag{A23}$$

We now differentiate this equation with respect to V

$$[\partial_\Gamma + (V - \Gamma)n_\Gamma] \partial_V \psi_\Gamma(V) = 0 \quad \text{for} \quad V \geq \Gamma \tag{A24}$$

$$\partial_\Gamma \partial_V \psi_\Gamma(V) = 2n_\Gamma (\psi_\Gamma(V) + \psi_\Gamma(\Gamma - V)) \quad \text{for} \quad 0 \leq V \leq \Gamma \tag{A25}$$

and keep the following condition at $V = \Gamma$

$$\partial_\Gamma \psi_\Gamma(V)|_{V=\Gamma} = 2n_\Gamma \int_0^\Gamma dV' \psi_\Gamma(V') \tag{A26}$$

The normalisation condition (A9) now reads

$$\int_0^F dW D_\Gamma^\pm(F, W) = n_\Gamma \int_0^\infty dll P_\Gamma^\pm(F, l) = -n_\Gamma \partial_p \left(U_\Gamma^\pm(p) e^{-(F-\Gamma)u_\Gamma^\pm(p)} \right)|_{p=0} \tag{A27}$$

$$= n_\Gamma u_\Gamma^\pm(0) e^{-(F-\Gamma)u_\Gamma^\pm(0)} \left(-\partial_p \ln U_\Gamma^\pm(p)|_{p=0} + (F - \Gamma)\partial_p u_\Gamma^\pm(p)|_{p=0} \right) \tag{A28}$$

Taking into account the changes of variables

$$\int_0^F dW D_\Gamma^\pm(F, W) = n_\Gamma u_\Gamma^\pm(0) e^{-(F-\Gamma)u_\Gamma^\pm(0)} \int_0^F dW \Psi_\Gamma^\pm(F - W, W) \tag{A29}$$

$$= 2n_\Gamma u_\Gamma^\pm(0) e^{-(F-\Gamma)u_\Gamma^\pm(0)} \int_0^F dW \psi_\Gamma(W) \tag{A30}$$

we get for $F \geq \Gamma$

$$2 \int_0^F dW \psi_\Gamma(W) = -\partial_p \ln U_\Gamma^\pm(p)|_{p=0} + (F - \Gamma)\partial_p \ln u_\Gamma^\pm(p)|_{p=0} \tag{A31}$$

Deriving with respect to F, we get for any $F \geq \Gamma$

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$$\psi_\Gamma(F \geq \Gamma) = \psi_\Gamma(\Gamma) = \frac{1}{2}\partial_p u_\Gamma^\pm(p)|_{p=0} = \frac{(\sinh(2\delta\Gamma) - 2\delta\Gamma)}{8\delta \sinh^2(\delta\Gamma)} \tag{A32}$$

For the value $F = \Gamma$ we get

$$\int_0^\Gamma dW\,\psi_\Gamma(W) = -\frac{1}{2}\partial_p \ln U_\Gamma^\pm(p)|_{p=0} = \frac{\delta\Gamma \coth(\delta\Gamma) - 1}{4\delta^2} \tag{A33}$$

The equation (A25) for $V \geq \Gamma$ is thus satisfied, as well as the condition (A26) at $V = \Gamma$. We now have to find the solution $\psi_\Gamma(V)$ to the equation (A25) on the domain $0 \leq V \leq \Gamma$ that satisfies the continuity boundary conditions $\psi_\Gamma(V = \Gamma) = \psi_\Gamma(\Gamma)$ given in (A32) and $\partial_V \psi_\Gamma(V)|_{V=\Gamma} = 0$. It reads

$$\psi_\Gamma(V) = \psi_\Gamma(\Gamma) - \frac{(\sinh(2\delta(\Gamma - V)) - 2\delta(\Gamma - V))}{4\delta \sinh^2(\delta\Gamma)} \tag{A34}$$

The final result is thus that the joint distributions $D_\Gamma^\pm(F, W)$ are given by

$$D_\Gamma^\pm(F, W) = n_\Gamma u_\Gamma^\pm e^{-(F-\Gamma)u_\Gamma^\pm}\left(\psi_\Gamma(W) + \psi_\Gamma(F - W)\right) \tag{A35}$$

where

$$\psi_\Gamma(W) = \frac{1}{2}h_\Gamma(\Gamma) - \theta(\Gamma - W)h_\Gamma(\Gamma - W) \tag{A36}$$

in terms of the auxiliary function

$$h_\Gamma(V) = \frac{(\sinh(2\delta V) - 2\delta V)}{4\delta \sinh^2(\delta\Gamma)} \tag{A37}$$

We now compute the distributions of W alone

$$D_\Gamma^\pm(W) = \int_{\max(\Gamma, W)}^{+\infty} dF\,D_\Gamma^\pm(F, W) \tag{A38}$$

$$= n_\Gamma\left(h_\Gamma(\Gamma) - \theta(\Gamma - W)h_\Gamma(\Gamma - W)\right)e^{-\max(0, W-\Gamma)u_\Gamma^\pm} - n_\Gamma \int_0^{\min(\Gamma, W)} dV\,u_\Gamma^\pm e^{(V-W)u_\Gamma^\pm}h_\Gamma(V) \tag{A39}$$

More explicitly, for $W \geq \Gamma$, we have a simple exponential form in W

$$D_\Gamma^\pm(W > \Gamma) = e^{-(W-\Gamma)u_\Gamma^\pm}n_\Gamma\left[h_\Gamma(\Gamma) - u_\Gamma^\pm\int_0^\Gamma dV\,e^{-(\Gamma-V)u_\Gamma^\pm}h_\Gamma(V)\right) \tag{A40}$$

whereas for $0 \leq W \leq \Gamma$, we have

$$D_\Gamma^\pm(W < \Gamma) = n_\Gamma\left(h_\Gamma(\Gamma) - h_\Gamma(\Gamma - W) - u_\Gamma^\pm\int_0^W dV\,e^{(V-W)u_\Gamma^\pm}h_\Gamma(V)\right) \tag{A41}$$

These expressions lead to the final results (49) and (50) given in the text.

3. Joint distribution of position and energy

The path-integral approach to compute the joint distribution of position and energy can be generalized to the biased case as follows.

The basic path-integral (55) becomes

$$F_{[V_a, V_b]}^\pm(V_0, x_0, V, x) = \int_{V_{x_0}=V_0}^{V_l=V} DV(y)\,exp\left(-\int_0^x dy\left(\frac{1}{4}\left(\frac{dV}{dy} \mp 2\delta\right)^2\right)\theta_{[V_a, V_b]}^-(\{V\}) \tag{A42}$$

$$= e^{-\delta^2(x-x_0)\pm\delta(V-V_0)}F_{[V_a, V_b]}(V_0, x_0, V, x) \tag{A43}$$

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and thus in Laplace transform with respect to $(x - x_0)$, we have in terms of (57)

$$\hat{F}^{\pm}_{[V_a,V_b]}(V,p|V_0) = e^{\pm\delta(V-V_0)}\hat{F}_{[V_a,V_b]}(V,p+\delta^2|V_0) \tag{A44}$$

Similarly, (64) and (65) become

$$\hat{B}^{\pm}_{\Gamma}(F,p) = e^{\pm\delta\Gamma}\frac{\sqrt{p+\delta^2}}{\sinh\sqrt{p+\delta^2}\Gamma}e^{-(F-\Gamma)\left(\mp\delta+\sqrt{p+\delta^2}\coth\sqrt{p+\delta^2}\Gamma\right)} \tag{A45}$$

$$\hat{E}_{\Gamma}(V,p|V_0) = e^{-(V-V_0)\left(\mp\delta+\sqrt{p+\delta^2}\coth\sqrt{p+\delta^2}\Gamma\right)} \tag{A46}$$

<div align="center">

a.   Case $W > \Gamma$

</div>

Using the same decomposition as in the symmetric case (see Figure 3 and (71)), the path-integral (112) becomes

$$\hat{P}^{\pm}_t(x,W) = \frac{\delta^2}{\sinh^2\delta\Gamma}\int_W^{W+\Gamma}dV_m\int_0^x dx_m B^{\pm}_{\Gamma}(V_m,x_m)\left(\partial_{V_0}F^{\pm}_{[V_m-\Gamma,V_m]}(W,x|V_0,x_m)\right)|_{V_0=V_m} \tag{A47}$$

$$\int_x^{+\infty}dx'_m\left(\partial_V F^{\pm}_{[V_m-\Gamma,V_m]}(W,x'_m|V,x)\right)|_{V=V_m}\int_{V_m}^{+\infty}dF\int_{x'_m}^{+\infty}dl\,E^{\pm}_{\Gamma}(F,l;V_m,x'_m) \tag{A48}$$

and thus we obtain in Laplace transform

$$\hat{P}^{\pm}_t(p,W) = \frac{\delta^2}{\sinh^2\delta\Gamma}\int_W^{W+\Gamma}dV_m e^{\pm\delta\Gamma}\frac{\sqrt{p+\delta^2}}{\sinh\sqrt{p+\delta^2}\Gamma}e^{-(V_m-\Gamma)(\mp\delta+\sqrt{p+\delta^2}\coth\sqrt{p+\delta^2}\Gamma)} \tag{A49}$$

$$\frac{\sinh\sqrt{p+\delta^2}(\Gamma-(V_m-W))}{\sinh\sqrt{p+\delta^2}\Gamma}\frac{\sinh\delta(\Gamma-(V_m-W))}{\sinh\delta\Gamma}\int_{V_m}^{+\infty}dF e^{-(F-V_m)(\mp\delta+\delta\coth\delta\Gamma)} \tag{A50}$$

which leads to the final result (113) given in the text.

<div align="center">

b.   Case $W < \Gamma$

</div>

As in the symmetric case, there are two contributions. The first one corresponding to the case where $V_m > \Gamma$ is the same as (A48) except that now we have to integrate over $V_m \in (\Gamma, W+\Gamma)$ (instead of $V_m \in (W, W+\Gamma)$) and we get

$$\hat{P}^{(1)\pm}_t(p,W<\Gamma) = \frac{\delta^2}{\sinh^2\delta\Gamma}\int_\Gamma^{W+\Gamma}dV_m e^{\mp\delta\Gamma}\frac{\sqrt{p+\delta^2}}{\sinh\sqrt{p+\delta^2}\Gamma}e^{-(V_m-\Gamma)(\pm\delta+\sqrt{p+\delta^2}\coth\sqrt{p+\delta^2}\Gamma)} \tag{A51}$$

$$\frac{\sinh\sqrt{p+\delta^2}(\Gamma-(V_m-W))}{\sinh\sqrt{p+\delta^2}\Gamma}\frac{\sinh\delta(\Gamma-(V_m-W))}{\sinh\delta\Gamma}\int_{V_m}^{+\infty}dF e^{-(F-V_m)(\pm\delta+\delta\coth\delta\Gamma)} \tag{A52}$$

$$=\frac{\delta^2}{\sinh^2\delta\Gamma\left(\frac{p+\delta^2}{\sinh^2\sqrt{p+\delta^2}\Gamma}\mp 4\delta(\mp\delta+\sqrt{p+\delta^2}\coth\sqrt{p+\delta^2}\Gamma)\right)} \tag{A53}$$

$$[-2(\mp\delta+\sqrt{p+\delta^2}\coth\sqrt{p+\delta^2}\Gamma)e^{-W(\mp\delta+\sqrt{p+\delta^2}\coth\sqrt{p+\delta^2}\Gamma)} \tag{A54}$$

$$+2(\mp\delta+\sqrt{p+\delta^2}\coth\sqrt{p+\delta^2}\Gamma)\cosh\sqrt{p+\delta^2}W\cosh\delta W \tag{A55}$$

$$-(\sqrt{p+\delta^2}(1+\coth^2\sqrt{p+\delta^2}\Gamma)\mp 2\delta\coth\sqrt{p+\delta^2}\Gamma))\sinh\sqrt{p+\delta^2}W\cosh\delta W \tag{A56}$$

$$+\frac{1}{\delta}(p-(\mp\delta+\sqrt{p+\delta^2}\coth\sqrt{p+\delta^2}\Gamma)^2)\cosh\sqrt{p+\delta^2}W\sinh\delta W \tag{A57}$$

$$+(\mp\delta+\sqrt{p+\delta^2}\coth\sqrt{p+\delta^2}\Gamma)(-2\coth\sqrt{p+\delta^2}\Gamma+\frac{\sqrt{p+\delta^2}}{\delta\sinh^2\sqrt{p+\delta^2}\Gamma})\sinh\sqrt{p+\delta^2}W\sinh\delta W] \tag{A58}$$

The second contribution corresponding to the case where $V_m < \Gamma$ (see Figure (4) can be obtained as in (88)

<div align="center">20</div>

$$\hat{P}_t^{(2)\pm}(x, W < \Gamma) = \frac{\delta^2}{\sinh^2 \delta\Gamma} \left(\partial_{V_0} F_{[0,\Gamma]}^{\pm}(V_0, 0; W, x) \right)_{V_0=0} \int_x^{\infty} dz \left(\partial_{V'} F_{[0,\Gamma]}^{\pm}(W, x; , V', z) \right)_{V'=\Gamma} \tag{A59}$$

$$\int_{\Gamma}^{+\infty} dF \int_z^{+\infty} dl E_{\Gamma}^{\pm}(\Gamma, z; F, l) \tag{A60}$$

and thus in Laplace transform

$$\hat{P}_t^{(2)\pm}(p, W) = \frac{\delta^2}{\sinh^2 \delta\Gamma} \left(\partial_{V_0} F_{[0,\Gamma]}^{\pm}(w, p|V_0) \right)_{V_0=0} \int_x^{\infty} dz \left(\partial_{V'} F_{[0,\Gamma]}^{\pm}(V', 0|W) \right)_{V'=\Gamma} \tag{A61}$$

$$\int_{\Gamma}^{+\infty} dF E_{\Gamma}^{\pm}(F, 0|\Gamma) \tag{A62}$$

$$= \frac{\delta \sinh \delta W \sinh(\Gamma - W) \sqrt{p + \delta^2}}{\sinh^2 \delta\Gamma \sinh \Gamma \sqrt{p + \delta^2}} \tag{A63}$$

The final result for $W < \Gamma$ is obtained by summing the two contributions (A58) and (A63) and is given in (120) in the text.

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